

On a bounded version of Hölder's Theorem and its application to the permutability equation

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This chapter is dedicated to Patrick Suppes, whose works and counsel have shaped much of my scientific life.

Abstract

The permutability equation $G(G(x, y), z) = G(G(x, z), y)$ is satisfied by many scientific and geometric laws. A few examples among many are: The Lorentz-FitzGerald Contraction, Beer's Law, the Pythagorean Theorem, and the formula for computing the volume of a cylinder. We prove here a representation theorem for the permutability equation, which generalizes a well-known result. The proof is based on a bounded version of Hölder's Theorem.

Hölder's Theorem on ordered groups is a foundation stone of measurement theory (c.f. Krantz et al., 1971; Suppes et al., 1989; Luce et al., 1990), and so, of much of quantitative science. There are several renditions of it. Whatever the version, the theorem concerns an algebraic structure $(\mathcal{X}, \circ, \preceq)$, in which \mathcal{X} is a set, \circ is an operation on \mathcal{X} , and \preceq is a weak order on \mathcal{X} (transitive, connected), which may be a simple order (antisymmetric). The axioms imply the existence of a function $f : \mathcal{X} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} x \preceq y &\iff f(x) \leq f(y) \\ f(x \circ y) &= f(x) + f(y) \quad (\text{whenever } x \circ y \text{ is defined}). \end{aligned}$$

Most formulations of this theorem have one or both of two drawbacks.

HYPOTHESIS 1. The elements of \mathcal{X} can be arbitrarily large.

HYPOTHESIS 2. The elements of \mathcal{X} can be arbitrarily small.

From the standpoint of social sciences applications, both of these hypotheses are unwarranted because the sensory mechanisms of humans and animals restrict the range of usable stimuli. In psychophysics, for example, small stimuli are undetectable by the sensory mechanisms, and large ones would damage them. Even in physics (relativity) the hypothesis that infinitely large quantities exist is inconsistent with current theories. In the axiomatization of Luce and Marley (1969) (see also Krantz et al., 1971, page 84), arbitrarily large elements need not exist. However, they use the following solvability axiom, which essentially asserts the existence of arbitrarily small elements.

If $x \prec y$, then there is some z such that $x \circ z \preceq y$.

It might be argued that these two hypotheses are idealizations, and that using them simplifies the derivations. The trouble is that, in the framework of the other axioms, these two hypotheses imply that the operation \circ is commutative. But commutativity is an essential property, which is testable empirically. To derive such a property from questionable axioms is not ideal. Our Lemma 7 is a version of Hölder's Theorem, due to Falmagne (1975), in which neither arbitrarily small, nor arbitrarily large elements are assumed to exist, and in which commutativity is an independent axiom.

We use this lemma to prove a representation theorem for the 'permutability' property, which is an abstract constraint on a real, positive valued function G of two real positive variables. This property is formalized by the equation

$$G(G(y, r), t) = G(G(y, t), r), \quad (1)$$

where G is strictly monotonic and continuous in both variables. An interpretation of $G(y, r)$ in Equation (1) is that the second variable r modifies the state of the first variable y , creating an effect evaluated by $G(y, r)$ in the same measurement variable as y . The left hand side of (1) represents a one-step iteration of this phenomenon, in that $G(y, r)$ is then modified by t , resulting in the effect $G(G(y, r), t)$. Equation (1), which is referred to as the 'permutability' condition in the functional equations literature (c.f. Aczél, 1966), formalizes the concept that the order of the two modifiers r and t is irrelevant. The importance of that property for scientific applications is that it can sometimes be inferred from a *gedanken* experiment, before any experimentation, thereby substantially constraining the possible models for a situation.

Indeed, under fairly general conditions of continuity and solvability making empirical sense, the permutability condition (1) implies the existence of a general representation

$$G(y, r) = f^{-1}(f(y) + g(r)), \quad (2)$$

where f and g are real valued, strictly monotonic continuous functions. We prove this fact here in the form of our Theorem 9, generalizing results of Hosszú (1962a,b,c) (cf. also Aczél, 1966). It is easily shown that the representation (2) implies the permutability condition (1): we have

$$\begin{aligned} G(G(y, r), t) &= f^{-1}(f^{-1}(f(G(y, r)) + g(t))) && \text{(by (2))} \\ &= f^{-1}(f^{-1}(f(f^{-1}(f(y) + g(r))) + g(t))) && \text{(by (2) again)} \\ &= f^{-1}(f(y) + g(r) + g(t)) && \text{(simplifying)} \\ &= f^{-1}(f(y) + g(t) + g(r)) && \text{(by commutativity)} \\ &= G(G(y, t), r) && \text{(by symmetry).} \end{aligned}$$

We will also use a more general condition, called 'quasi permutability', which is defined by the equation

$$M(G(y, r), t) = M(G(y, t), r) \quad (3)$$

and lead to the representation

$$M(y, r) = m((f(y) + g(r))). \quad (4)$$

In our first section, we state some basic definitions and we describe a few examples of laws, taken from physics and geometry, in which the permutability condition applies. We also give one example, van der Waals Equation, which is not permutable. The second section is devoted to some preparatory lemmas. The last section contain the main results of the paper.

Basic Concepts and Examples

1 Definition. We write \mathbb{R}_+ and \mathbb{R}_{++} for the nonnegative and the positive reals, respectively. Let J , J' , and H be real nonempty and nonnegative intervals. A (*numerical*) *code* is a function $M : J \times J' \xrightarrow{\text{onto}} H$ which is strictly increasing in the first variable, strictly monotonic in the second one, and continuous in both. A code M is *solvable* if it satisfies the following two conditions.

[S1] If $M(x, t) < p \in H$, there exists $w \in J$ such that $M(w, t) = p$.

[S2] The function M is *1-point right solvable*, that is, there exists a point $x_0 \in J$ such that for every $p \in H$, there is $v \in J'$ satisfying $M(x_0, v) = p$. In such a case, we may say that M is x_0 -solvable.

By the strict monotonicity of M , the points w and v of [S1] and [S2] are unique.

Two functions $M : J \times J' \rightarrow H$ and $G : J \times J' \rightarrow H'$ are *comonotonic* if

$$M(x, s) \leq M(y, t) \iff G(x, s) \leq G(y, t), \quad (x, y \in J; s, t \in J'). \quad (5)$$

In such a case, the equation

$$F(M(x, s)) = G(x, s) \quad (x \in J; s \in J') \quad (6)$$

defines a strictly increasing continuous function $F : H \xrightarrow{\text{onto}} H'$. We may say then that G is *F-comonotonic* with M .

We turn to the key condition of this paper.

2 Definition. A function $M : J \times J' \rightarrow H$ is *quasi permutable* if there exists a function $G : J \times J' \rightarrow J$ co-monotonic with M such that

$$M(G(x, s), t) = M(G(x, t), s) \quad (x, y \in J; s, t \in J'). \quad (7)$$

We say in such a case that M is *permutable with respect to G*, or *G-permutable* for short. When M is permutable with respect to itself, we simply say that M is *permutable*, a terminology consistent with Aczél (1966, Chapter 6, p. 270).

We mention the straightforward consequence:

3 Lemma. A function $M : J \times J' \rightarrow H$ is *G-permutable* only if G is permutable.

PROOF. Suppose that G is F -comonotonic with M . For any $x \in J$ and $s, t \in J'$, we get $G(G(x, s), t) = F(M(G(x, s), t)) = F(M(G(x, t), s)) = G(G(x, t), s)$. \square

Many scientific laws embody permutable or quasi permutable numerical codes, and hence can be written in the form of Equation (2). We give four quite different examples below. In each case, we derive the forms of the functions f and g in the representation equation (2).

4 Four Examples and One Counterexample.

(a) THE LORENTZ-FITZGERALD CONTRACTION. This term denotes a phenomenon in special relativity, according to which the apparent length of a rod measured by an observer moving at the speed v with respect to that rod is a decreasing function of v , vanishing as v approaches the speed of light. This function is specified by the formula

$$L(\ell, v) = \ell \sqrt{1 - \left(\frac{v}{c}\right)^2}, \quad (8)$$

in which $c > 0$ denotes the speed of light, ℓ is the actual length of the rod (for an observer at rest with respect to the rod), and $L : \mathbb{R}_+ \times [0, c[\xrightarrow{\text{onto}} \mathbb{R}_+$ is the length of the rod measured by the moving observer.

The function L is a permutable code. Indeed, L satisfies the strict monotonicity and continuity requirements, and we have

$$L(L(p, v), w) = p \left(1 - \left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}} \left(1 - \left(\frac{w}{c}\right)^2\right)^{-\frac{1}{2}} = L(L(p, w), v). \quad (9)$$

Solving the functional equation

$$\ell \sqrt{1 - \left(\frac{v}{c}\right)^2} = f^{-1}(f(\ell) + g(v)) \quad (10)$$

leads to the Pexider equation (c.f. Aczél, 1966, pages 141-165)

$$f(\ell y) = f(\ell) + k(y) \quad (11)$$

$$\text{with} \quad k(y) = g\left(c\sqrt{1 - y^2}\right).$$

As the background conditions (monotonicity and domains of the functions¹) are satisfied, the unique forms of f and g in (11) are determined. They are: with $\xi > 0$,

$$f(\ell) = \xi \ln \ell + \theta \quad (12)$$

$$g(v) = \xi \ln \left(\sqrt{1 - \left(\frac{v}{c}\right)^2} \right). \quad (13)$$

¹Note that the standard solutions for Pexider equations are valid when the domain of the equation is an open connected subset of \mathbb{R}^2 rather than \mathbb{R}^2 itself. Indeed, Aczél (1987, see also Aczél, 2005, Chudziak and Tabor, 2008, and Radó and Baker, 1987) has shown that, in such cases, this equation can be extended to the real plane.

(b) **BEER'S LAW.** This law applies in a class of empirical situations where an incident radiation traverses some absorbing medium, so that only a fraction of the radiation goes through. In our notation, the expression of the law is

$$I(x, y) = x e^{-\frac{y}{c}}, \quad (x, y \in \mathbb{R}_+, c \in \mathbb{R}_{++} \text{ constant}) \quad (14)$$

in which x denotes the intensity of the incident light, y is the concentration of the absorbing medium, c is a reference level, and $I(x, y)$ is the intensity of the transmitted radiation. The form of this law is similar to that of the Lorentz-FitzGerald Contraction and the same arguments apply. Thus, the function $I : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+$ is also a permutable code. The solution of the functional equation

$$x e^{-\frac{y}{c}} = f^{-1}(f(x) + g(y))$$

follows a pattern identical to that of Equation (10) for the Lorentz-FitzGerald Contraction. The only difference lies in the definition of the function g , which is here

$$g(y) = -\xi \frac{y}{c}.$$

The definition of f is the same, namely (12). So, we get

$$I(x, y) = f^{-1}(f(x) + g(y)) = \exp \left(\frac{1}{\xi} (\xi \ln x + \theta - \xi \frac{y}{c} - \theta) \right) = x e^{-\frac{y}{c}}.$$

(c) **THE VOLUME OF A CYLINDER.** The permutability equation applies not only to many physical laws, but also to some fundamental formulas of geometry, such as the volume $C(\ell, r)$ of a cylinder of radius r and height ℓ , for example. In this case, we have

$$C(\ell, r) = \ell \pi r^2, \quad (15)$$

which is permutable. We have

$$C(C(\ell, r), v) = C(\ell \pi r^2, v) = \ell \pi r^2 \pi v^2 = C(C(\ell, v), r).$$

Solving the functional equation

$$\ell \pi r^2 = f^{-1}(f(\ell) + g(r))$$

yields the solution

$$f(\ell) = \xi \ln \ell + \theta$$

(again, the function f is the same as in the two preceding examples), and

$$g(r) = \xi \ln (\pi r^2),$$

with

$$f^{-1}(f(\ell) + g(r)) = \exp \left(\frac{1}{\xi} (\xi \ln \ell + \theta + \xi \ln (\pi r^2) - \theta) \right) = \ell \pi r^2.$$

We give another geometric example below, in which the form of f is different.

(d) THE PYTHAGOREAN THEOREM. The function

$$P(x, y) = \sqrt{x^2 + y^2} \quad (x, y \in \mathbb{R}_{++}), \quad (16)$$

representing the length of the hypotenuse of a right triangle in terms of the lengths of its sides, is a permutable code. We have indeed

$$P(P(x, y), z) = \sqrt{P(x, y)^2 + z^2} = \sqrt{x^2 + y^2 + z^2} = P(P(x, z), y).$$

The function P is symmetric. So we must solve the equation

$$\sqrt{x^2 + y^2} = f^{-1}(f(x) + f(y))$$

or, equivalently,

$$f\left(\sqrt{x^2 + y^2}\right) = f(x) + f(y). \quad (17)$$

With $z = x^2$, $w = y^2$, and defining the function $h(z) = f\left(z^{\frac{1}{2}}\right)$, Equation (17) becomes

$$h(z + w) = h(z) + h(w),$$

a Cauchy equation on the positive reals, with h strictly increasing. It has the unique solution $h(z) = \xi z$, for some positive real number ξ (c.f. Aczél, 1966, page 31). We get

$$f(x) = \xi x^2$$

and

$$f^{-1}(f(x) + f(y)) = \left(\frac{1}{\xi}(\xi x^2 + \xi y^2)\right)^{\frac{1}{2}} = \sqrt{x^2 + y^2}.$$

(f) THE COUNTEREXAMPLE: VAN DER WAALS EQUATION. One form of this equation is

$$T(p, v) = K \left(p + \frac{a}{v^2}\right) (v - b), \quad (18)$$

in which p is the pressure of a fluid, v is the volume of the container, T is the temperature, and a , b and K are constants; K is the reciprocal of the Boltzmann constant. It is easily shown that the function T in (18) is not permutable.

5 Open Problem. Examining the four examples (a) to (d) above suggests that once the exact form of a permutable law is known, the form of the functions f and g in the representation (2) can easily be guessed. For example, in each of the problems (a), (b), and (c), the permutable law is the product of two functions, with the first one being the identity function. In these problems, the form of f is the same, namely $f(x) = \xi \ln x + \theta$. A more difficult problem is: are there basic structural properties which, in addition to permutability, determine the form of a permutable law, possibly up to some parameters? We will consider this problem in a later paper.

Preparatory Results

The main step in our developments is based on the following construction.

6 Definition. Suppose that $G : J \times J' \rightarrow J$ is a code that is x_0 -solvable in the sense of Condition [S2]. Define the operation \bullet on J by the equivalence

$$x \bullet y = G(x, v) \iff G(x_0, v) = y \quad (x, y \in J; v \in J'). \quad (19)$$

We show in this section that a solvable code G is permutable if and only if it has an additive representation

$$G(y, v) = f^{-1}(f(y) + g(v)) \quad (x, y \in J; v \in J') \quad (20)$$

where $f : J \rightarrow \mathbb{R}_+$ and $g : J' \rightarrow \mathbb{R}_+$ are continuous functions with f strictly increasing and g strictly monotonic.

Our basic tool lies in the following lemma.

7 Lemma. Let J be a real non degenerate interval. With $R \subseteq J \times J$, let $\bullet : R \rightarrow J$ be a non necessarily closed operation on J . We write xRy to mean that $x \bullet y$ is defined. Suppose that the triple (J, \bullet, \leq) , where \leq is the inequality of the reals, satisfies the following five independent conditions:

- (i) yRx if xRy , and when yRx , then $y \bullet x = x \bullet y$;
- (ii) whenever yRx , wRz , wRy' , $z'Rx$, yRy' and $z'Rz$, then

$$(y \bullet x = w \bullet z) \text{ and } (w \bullet y' = z' \bullet x) \implies y \bullet y' = z' \bullet z;$$

- (iii) there exists $x \in J$ such that xRx and $x \bullet xRx$;
- (iv) if $y \bullet x < z$, then $y \bullet w = z$ for some w in J ;
- (v) for every x, y and z in J , with $x < y$, the set $N(x, z; y) = \{n \in \mathbb{N}^+ \mid x_y^n \leq z\}$ is finite, where the sequence (x_y^n) is defined recursively as follows:

- (a) $x_y^1 = x$;
- (b) if x_y^{n-1} is defined and x' exists such that $y \bullet x_y^{n-1} = x \bullet x'$ then $x_y^n = x'$.

Then, there exists a strictly increasing function $f : J \rightarrow J$ such that

$$f(x \bullet y) = f(y) + f(y).$$

(For a proof, see Falmagne, 1975).

8 Lemma. Let $G : J \times J' \rightarrow J$ be a solvable, permutable code. Then, the triple (J, \bullet, \leq) , with the operation \bullet defined by (19), satisfies Conditions (i)-(v) of Lemma 7. Moreover, the operation \bullet is associative, strictly increasing and continuous in both variables.

PROOF. Take any $x, y \in J$ with

$$G(x_0, r) = x \quad (21)$$

and

$$G(x_0, v) = y. \quad (22)$$

(i) By (19), (21), (22) and the permutability of G , we get successively,

$$y \bullet x = G(y, r) = G(G(x_0, v), r) = G(G(x_0, r), v) = G(x, v) = x \bullet y.$$

(ii) Suppose that

$$(y \bullet x = w \bullet z) \text{ and } (w \bullet y' = z' \bullet x). \quad (23)$$

With (21), (22) and

$$G(x_0, s) = z, \quad G(x_0, t) = w, \quad G(x_0, v') = y', \quad G(x_0, s') = z', \quad (24)$$

we get from (23)

$$G(y, r) = G(w, s) \quad (25)$$

$$G(w, v') = G(z', r). \quad (26)$$

Equation (25) gives

$$G(G(y, r), v') = G(G(w, s), v'),$$

which yields successively

$$\begin{aligned} G(G(y, v'), r) &= G(G(w, v'), s) && \text{(by permutability)} \\ &= G(G(z', r), s) && \text{(by (26))} \\ &= G(G(z', s), r) && \text{(by permutability),} \end{aligned}$$

so

$$G(G(y, v'), r) = G(G(z', s), r).$$

By the strict monotonicity of G in the first variable, we obtain $G(y, v') = G(z', s)$ and thus $y \bullet y' = z' \bullet z$.

(iii) By the solvability condition [S2], there exists $x \in J$ such that, with $G(x_0, r) = x$, we have both

$$x \bullet x = G(x, r) \in J \quad \text{and} \quad (x \bullet x) \bullet x = G(G(x, r), r) \in J.$$

(iv) If $x \bullet y < z$, then $y \bullet x = G(y, r) < z \in J$ by commutativity, (21), and the definition of \bullet . Applying [S1], we get $G(w, r) = z$ for some $w \in J$. Using again (21), we obtain $x \bullet w = z$.

(v) We first show that the sequence (x_y^n) defined by (a) and (b) is strictly increasing. We proceed by induction. Since $x < y$ by definition, we get from (21) and (22)

$$x = G(x_0, r) < G(x_0, v) = y,$$

with the function G strictly monotonic in its second variable. In the sequel, we suppose that G is strictly decreasing in its second variable; so,

$$v < r. \quad (27)$$

The proof is similar in the other case. The following equalities hold by the definitions of x_y^1 , x_y^2 and commutativity:

$$y \bullet x_y^1 = y \bullet x = G(y, r) = x \bullet y = x \bullet x_y^2 = x_y^2 \bullet x = G(x_y^2, r).$$

From $G(y, r) = G(x_y^2, r)$, we get $x_y^2 = y$ and $x_y^1 < x_y^2$. Assuming that $x_y^{n-1} < x_y^n$, we get $y \bullet x_y^n = x \bullet x_y^{n+1}$ by the definition of the term x_y^{n+1} in Condition (v) (b) of Lemma 7, and by commutativity

$$x_y^n \bullet y = G(x_y^n, v) = x_y^{n+1} \bullet x = G(x_y^{n+1}, r),$$

yielding $G(x_y^n, v) = G(x_y^{n+1}, r)$. Since $v < r$ and G is decreasing in its second variable

$$G(x_y^{n+1}, v) > G(x_y^{n+1}, r) = G(x_y^n, v),$$

and so

$$x_y^n < x_y^{n+1}$$

because G is strictly increasing in its first variable. By induction, the sequence (x_y^n) is strictly increasing.

Suppose that the set $N(x, z; y)$ of Condition (v) is not finite. Thus, the point z is an upper bound of the sequence (x_y^n) . Because this sequence is increasing and bounded above, it necessarily converges. Without loss of generality, we can assume that we have in fact $\lim_{n \rightarrow \infty} x_y^n = z$. Since

$$y \bullet x_y^{n-1} = x \bullet x_y^n < x \bullet z$$

for all $n \in \mathbb{N}$, the solvability Condition (iv) implies that there is some $z' \in J$ such that $y \bullet z' = x \bullet z$, with necessarily $z' < z$. There must be some $m \in \mathbb{N}$ such that $z' < x_y^m < z$. We obtain thus

$$x \bullet z = y \bullet z' < y \bullet x_y^m = x \bullet x_y^{m+1}$$

and so $z < x_y^{m+1}$, in contradiction with $\lim_{n \rightarrow \infty} x_y^n = z$, with (x_y^n) an increasing sequence. We conclude that the set $N(x, z; y)$ must be finite for all x, y and z in J , with $x < y$. We conclude that the Conditions (i)-(v) of Lemma 7 are satisfied.

To prove that \bullet is associative, we take any x, y and z in J . Using again $G(x_0, r) = x$, $G(x_0, v) = y$ and $G(x_0, s) = z$, we have

$$\begin{aligned} x \bullet (y \bullet z) &= G(y \bullet z, r) && (\text{since } G(x_0, r) = x) \\ &= G(G(y, s), r) && (\text{since } G(x_0, s) = z) \\ &= G(G(y, r), s) && (\text{by permutability}) \\ &= G(x \bullet y, s) && (\text{since } G(x_0, r) = x) \\ &= z \bullet (x \bullet y) && (\text{since } G(x_0, s) = z) \\ &= (x \bullet y) \bullet z && (\text{by commutativity}). \end{aligned}$$

Finally, since for all $x, y \in J$, we have

$$x \bullet y = G(y, r) = y \bullet x = G(x, v),$$

it is clear that the operation \bullet is continuous and strictly increasing in both variables. \square

Main Result

The theorem below generalizes results of Hosszú (1962a,b,c) (cf. also Aczél, 1966).

9 Theorem. (i) *A solvable code $M : J \times J' \rightarrow H$ is quasi permutable if and only if there exists three continuous functions $m : \{f(y) + g(r) \mid x \in J, r \in J'\} \rightarrow H$, $f : J \rightarrow \mathbb{R}$, and $g : J' \rightarrow \mathbb{R}$, with m and f strictly increasing and g strictly monotonic, such that*

$$M(y, r) = m(f(y) + g(r)). \quad (28)$$

(ii) *A solvable code $G : J \times J' \rightarrow J$ is a permutable code if and only if, with f and g as above, we have*

$$G(y, r) = f^{-1}(f(y) + g(r)). \quad (29)$$

(iii) *If a solvable code $G : J \times J \rightarrow J$ is a symmetric function—that is, $G(x, y) = G(y, x)$ for all $x, y \in J$ —then G is permutable if and only if there exists a strictly increasing and continuous function $f : J \rightarrow J$ satisfying*

$$G(x, y) = f^{-1}(f(x) + f(y)). \quad (30)$$

(iv) *If the code G in (29) is differentiable in both variables, with non vanishing derivatives, then the functions f and g are differentiable. This differentiability result also applies to the code G and the function f in (30).*

Our argument for establishing (i) and (ii) is essentially the same as that in Aczél (1966, p. 271-273) but, because our solvability conditions [S1]-[S2] are weaker, it relies on Lemma 7 rather than on the representation in the reals of an ordered Archimedean group (for example, c.f. Hölder, 1901).

PROOF. (i)-(ii) Suppose that the code M of the theorem is permutable with respect to a F -comonotonic code G . By Lemma 3, the code G is permutable. Defining the operation $\bullet : J \times J' \rightarrow J$ by

$$y \bullet x = G(y, r) \iff G(x_0, r) = x, \quad (31)$$

it follows from Lemma 8 that the triple (J, \bullet, \leq) satisfies Conditions (i)-(v) of Lemma 7, with the operation \bullet associative and continuously increasing in both variable. Accordingly, there exists a continuous, strictly increasing function $f : J \rightarrow J$ such that

$$f(y \bullet x) = f(y) + f(x). \quad (32)$$

Defining the strictly monotonic function $\psi : J' \rightarrow J$ by

$$\psi(s) = G(x_0, s),$$

we get from (31) and (32),

$$f(y \bullet x) = f(G(y, r)) = f(y \bullet G(x_0, r)) = f(y) + f(\psi(r)),$$

and thus

$$G(y, r) = f^{-1}(f(y) + f(\psi(r))),$$

or with $g = f \circ \psi$,

$$G(y, r) = f^{-1}(f(y) + g(r)). \quad (33)$$

(Notice that $f(y) + g(r) \in J$.) Because G is F -comonotonic with M , and F maps H onto J , we obtain

$$M(y, r) = F^{-1}(G(y, r)) = (F^{-1} \circ f^{-1})(f(y) + g(r)),$$

or, with $m = F^{-1} \circ f^{-1}$,

$$M(y, r) = m(f(y) + g(r)) \quad (y \in J; r \in J'; f(y) + g(r) \in J). \quad (34)$$

It is clear that the functions f and g in (33) and the functions m , f and g in (34) are continuous, with the required monotonicity properties. This proves the necessity part of (i). The sufficiency is straightforward.

(ii) This was established in passing: cf. Eq. (33).

(iii) From (ii), we get by the symmetry of G

$$G(x, y) = f^{-1}(f(x) + g(y)) = G(y, x) = f^{-1}(f(y) + g(x))$$

yielding

$$f(x) - g(x) = f(y) - g(y) = K$$

for some constant K and all $x, y \in J$. We have thus $g(x) = f(x) - K$ for all $x \in J$. Since $g^{-1}(t) = f^{-1}(t + K)$, we obtain

$$g^{-1}(g(x) + g(y)) = f^{-1}(g(x) + g(y) + K) = f^{-1}(f(x) + g(y)) = G(x, y).$$

Defining $h = g$, we obtain (30).

(iv) If the code G in (29) is differentiable with non vanishing derivatives, then, for every $r \in J$, the inverse G_r^{-1} of G in the first variable is differentiable. From (29), we get $f(x) = G_r^{-1}(x) + g(r)$ with $G_r^{-1}(x) = y$. So, f is differentiable, and since, from (29) again,

$$g(r) = f(G(y, r)) - f(y)$$

with f differentiable and G differentiable in the second variable, g is also differentiable. The differentiability of f in (30) is immediate. \square

We mention in passing a simple uniqueness result concerning our basic representation equation (29).

10 Lemma. Suppose that the representation $G(y, r) = f^{-1}(f(y) + g(r))$ of Theorem 9(ii) holds for some code G , with f and g satisfying the stated continuity and monotonicity conditions. Then we also have $G(y, r) = (f^*)^{-1}(f^*(y) + g^*(r))$ for some continuous functions f^* and g^* , respectively co-monotonic with f and g , if and only if $f^* = \xi f + \theta$ and $g^* = \xi g$, for some constants $\xi > 0$ and θ .

PROOF. (Necessity.) Suppose that

$$(f^*)^{-1}(f^*(y) + g^*(r)) = f^{-1}(f(y) + g(r)).$$

Then, with $z = f(y)$ and $s = g(r)$ and applying f^* on both sides, we get

$$(f^* \circ f^{-1})(z) + (g^* \circ g^{-1})(s) = (f^* \circ f^{-1})(z + s), \quad (35)$$

a Pexider equation. It is clear that $(f^* \circ f^{-1})$ and $(g^* \circ g^{-1})$ are strictly increasing and continuous and that (35) is defined on an open connected subset of \mathbb{R}_+^2 . Accordingly (c.f. Aczél, 2005, and footnote 1), with $h = (f^* \circ f^{-1})$ and $k = m = (g^* \circ g^{-1})$, we get $(f^* \circ f^{-1})(z) = \xi z + \theta$ and $(g^* \circ g^{-1})(s) = \xi s$, $\xi > 0$, and so $f^*(y) = \xi f(y) + \theta$ and $g^*(r) = \xi g(r)$.

(Sufficiency.) If $f^* = \xi f + \theta$ and $g^* = \xi g$, with $\xi > 0$, then

$$\begin{aligned} (f^*)^{-1}(f^*(y) + g^*(r)) &= f^{-1} \left(\frac{f^*(y) + g^*(r) - \theta}{\xi} \right) \\ &= f^{-1} \left(\frac{\xi f(y) + \theta + \xi g(r) - \theta}{\xi} \right) \\ &= f^{-1}(f(y) + g(r)). \end{aligned}$$

□

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